Grade 6 Math Circles
February 13 & 14 & 15, 2024
Knights and Knaves - Problem Set Solutions

1. What values can a variable in Boolean algebra take?

> *Solution*: A variable in Boolean algebra can take the values true and false.

2. Pick one of the phrases below to complete the sentence: *In logic, the term "if and only if" between statements means that...*

   a. one of the two statements is true.

   b. both of the two statements are true.

   c. either the first statement is false and the second statement is true, or the first statement is true and the second statement is false.

   d. either both of the statements are true or both of the statements are false.

> *Solution*: The correct answer is d.

3. Let M be true and N be false. Assume that all variables in this question are boolean variables.

   (a) Let A represent the statement "Both M and N are true."
       What is the value of A?

   (b) Let B represent the statement "Either M is true or N is true."
       What is the value of B?

   (c) Let C represent the statement "Neither M nor N is true."
       What is the value of C?

   (d) Let D represent the statement "$M \iff N$."
       What is the value of D?

   (e) Let E represent the statement "$M \equiv N$."
       What is the value of E?

   (f) Let F represent the statement "M is not N."
       What is the value of F?

Solution:

(a) Since N is not true, then the statement "Both M and N are true" is false. Therefore, A is false.

(b) Since M is true, then the statement "Either M is true or N is true" is true. Therefore, B is true.

(c) Since M is true, then the statement "Neither M nor N is true" is false. Therefore, C is false.

(d) Since M is true whereas N is false, then the statement "$M \iff N$" is false. Therefore, D is false.

(e) Since M is true whereas N is false, then the statement "$M \equiv N$" is false. Therefore, E is false.

(f) Since M is true whereas N is false, then the statement "M is not N" is true. Therefore, F is false.

4. Let G represent the statement "Dakota has more berries than Quinn and Jayden is the same height as or taller than Elden." Suppose that G is false. What can we conclude? Avoid using the word *not* in your answer.

Solution: We can conclude that either Dakota has at most the same number of berries as Quinn or Jayden is shorter than Elden (or both).

5. Let H represent the statement "Everyone in Ms. Greenspan's class saw the 2023 film *Barbie*." Suppose that H is false. What can we conclude? You may use the word *not* in your answer if you wish.

Solution: We can conclude that someone in Ms. Greenspan's class did not see the 2023 film *Barbie*.

6. Let A, B, and C be Boolean variables. Complete the following truth table.

A	B	C	Exactly one of A and B are true	At least one of A, B, and C is true.
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

Solution:

A	B	C	Exactly one of A and B are true	At least one of A, B, and C is true.
T	T	T	F	T
T	T	F	F	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

7. In this question, we will solve the following Knights and Knaves problem by constructing a truth table.

> I meet two inhabitants of an island that consists of only Knights and Knaves. I ask one of them, "Is either of you a Knight?" Based on that person's response, I know my answer. What are each of the two inhabitants—are they Knights or Knaves?

(a) Let A represent the statement "The person who I ask the question to is a Knight."

Let B represent the statement "The other person I meet who I do not ask the question to is a Knight."

Fill in the following truth table.

A	B	Either of the two people I meet is a Knight	The person I ask responds "Yes"
T	T		
T	F		
F	T		
F	F		

Solution:

A	B	Either of the two people I meet is a Knight	The person I ask responds "Yes"
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

(b) Based on your truth table, what are each of the inhabitants? Explain your reasoning.

Solution: Looking at the truth table, there is only one response such that I can be certain of who is a Knight and who is a Knave. An answer of "No" reveals that the person I ask is a Knight and the other person is a Knave, whereas an answer of "Yes" is consistent with three possibilities.

Since the problem states that I am certain of the true answer to my question based on the response of the person who I pose the question to, then the person that I ask is a Knight and the other person is a Knave.

8. In this question, we will use a truth table to solve a Knights and Knave problem that involves three people. The problem is as follows.

> There are three people, Abby, Baloo, and Courtney, each of whom is either a Knight or a Knave. Abby and Baloo make the following statements:
>
>   Abby: All of us are Knaves.
>   Baloo: Exactly one of us is a Knight.
>
> What are Abby, Baloo, and Courtney?

(a) Let A represent the statement "Abby is a Knight."
Let B represent the statement "Baloo is a Knight."
Let C represent the statement "Courtney is a Knight."

Fill in the following truth table.

A	B	C	A, B, and C are all false	Exactly one of A, B, and C is true.
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

Solution:

A	B	C	A, B, and C are all false	Exactly one of A, B, and C is true.
T	T	T	F	F
T	T	F	F	F
T	F	T	F	F
T	F	F	F	T
F	T	T	F	F
F	T	F	F	T
F	F	T	F	T
F	F	F	T	F

(b) Based on your truth table, what are each of Abby, Baloo, and Courtney? Explain your reasoning.

Solution: Since Abby makes the statement "All of us are Knaves," then Abby is a Knight if and only if A, B, and C are all false.

Since Baloo makes the statement "Exactly one of us is a Knight," then Baloo is a Knight if and only if exactly one of A, B, C is true.

In other words, columns 1 and 4 must match and columns 2 and 5 must match in the correct solution. The sixth row (which is the third row from the bottom) in the table is the only row in which this is the case.

Therefore, Abby is a Knave, Baloo is a Knight, and Courtney is a Knave.

9. A barber of a certain small town shaved all inhabitants of the town who did not shave themselves, and never shaved any inhabitant who did shave themself. Explain why this situation is impossible. *Note: This is known as the Barber paradox.*

Solution: This situation is impossible because such a barber cannot possibly exist: The barber cannot shave themself, as the barber only shaves those who do *not* shave themselves.

But if the barber does not shave themself, then the barber is in the group of people who would be shaved by the barber, and thus the barber must shave themself.

10. Imagine that you are trying to find an island called Treasure Island. You know that you are on an island of only Knights and Knaves. You meet two inhabitants who make the following statements:

A: B is a Knight and this is Treasure Island.
B: A is a Knave and this is Treasure Island.

Is this Treasure Island? Justify your answer. You may use any method you wish.

Solution: We will solve this puzzle by constructing a truth table.

As usual, let A represent the statement that A is a Knight and B represent the statement that B is a Knight. Then we will construct a truth table with nested columns as follows.

A	B	B is a Knight and this is Treasure Island		A is a Knave and this is Treasure Island	
---	---	This is Treasure Island	This is not Treasure Island	This is Treasure Island	This is not Treasure Island
T	T	T	F	F	F
T	F	F	F	F	F
F	T	T	F	T	F
F	F	F	F	T	F

Since the statement that A makes is true if and only if A is a Knight and the statement that B makes is true if and only if B is a Knight, then we're looking for a row in which both columns 1 and 3 matches and columns 2 and 5 matches or in which both columns 1 and 4 matches and columns 2 and 6 matches (since the island could either be or not be Treasure Island).

If the island is Treasure Island, then none of the options work. However, if the island is not Treasure Island, then row 4 satisfies the conditions.

Therefore, A and B are both Knaves and the island is not Treasure Island.

11. Rama and Mysa are famous decorators of caskets. Whenever Rama decorates his caskets, he puts a true inscription on it, whereas Mysa puts only false inscriptions on his caskets. A jewel is in exactly one of three caskets with the inscriptions below, each decorated by either Rama or Mysa.

Casket 1: | The Jewel is in here. |
Casket 2: | The Jewel is in here. |
Casket 3: | At least two of these caskets were decorated by Mysa |

Determine the maker of each casket and choose the casket that contains the jewel. Justify your answer. You may use any method you wish.

Solution: The third casket must be Rama's. Suppose that it is not. Then it'd be decorated by Mysa, and the opposite of the inscription on it is true, and so at least two of the caskets were decorated by Rama. These caskets must be Casket 1 and 2, which is impossible since there is a jewel in one of the caskets.

Therefore, Caskets 1 and 2 were decorated by Rama and Casket 3 was decorated by Mysa. Since the jewel is not in Caskets 1 or 2, then the jewel must be located in Casket 3.